Permutations with Constrained Consecutive k-sums — Some Special Cases

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***Abstract*—For a permutation $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ of $1, 2, \ldots, n$ whose elements are written on a circle (and so indices are counted modulo n) we consider sums of all consecutive k elements, the difference $\mathrm{msum}(\pi, k)$ of maximal such sum and the average sum $k(n+1)/2$, and discrepancy $disc(\pi, k)$, the maximum distance of these sums from $k(n+1)/2$. Let $\mathrm{disc}(n, k)$ and $\mathrm{msum}(n, k)$ denote the minimum of $\mathrm{msum}(\pi, k)$ and $\mathrm{disc}(\pi, k)$, over all permutations $\pi \in S_n$. Here we prove that $\mathrm{disc}(6t + 3, 3) = 2$, completing previously known results about $\mathrm{disc}(n, 3)$. The value of $\mathrm{msum}(2km \pm 2, 2k) = 1$ is determined by supplying a construction.**

***Index Terms*—permutation, discrepancy, maximal consecutive sum.**

I. Introduction

Let $\pi = (\pi_1, \pi_2, \ldots, \pi_n) \in S_n$ and suppose $\pi_{i+n} = \pi_i$, $i \geq 1$. As in [1] consider subsequent k-sums $s_i = \sum_{j=1}^{k} \pi_{i+j}$, $i \geq 1$, their normalized maximum

$$\mathrm{msum}(\pi, k) = \max_{1 \leq i \leq n} s_i - k(n+1)/2 \tag{1}$$

and their *discrepancy*

$$\mathrm{disc}(\pi, k) = \max_{1 \leq i \leq n} |s_i - k(n+1)/2|; \tag{2}$$

here $k(n+1)/2 = \frac{1}{n} \sum_{i=1}^{n} s_i$ is the average of all subsequent k-sums. Furthermore, let

$$\mathrm{msum}(n, k) = \min_{\pi \in S_n} \mathrm{msum}(\pi, k) \tag{3}$$

and let

$$\mathrm{disc}(n, k) = \min_{\pi \in S_n} \mathrm{disc}(\pi, k). \tag{4}$$

Obviously, $\mathrm{msum}(\pi, k) \leq \mathrm{disc}(\pi, k)$ and $\mathrm{msum}(n, k) \leq \mathrm{disc}(n, k)$.

For example, if $k = 4$, $n = 10$ and $\pi \in S_{10}$ is the permutation

i	1	2	3	4	5	6	7	8	9	10
π_i	9	2	7	4	10	1	8	3	6	5
s_i	22	23	22	23	22	18	22	23	22	23

then $\mathrm{msum}(\pi, 4) = 23 - 22 = 1$.

Anstee et all. [1] derived some upper and lower bounds on $\mathrm{disc}(n, k)$. Given a permutation $\pi \in S_n$, they consider the vector $d = (d_1, d_2, \ldots, d_n)$ whose components are the differences

$$d_i = s_{i+1} - s_i = \pi_{i+k} - \pi_i, \quad i \geq 1.$$

If $1 \leq s \leq t \leq n$ then $\sum_{j=s}^{t} d_j = s_{t+1} - s_s$ is a diffference of some two k-sums. Nearly all discrepancies are determined with $k = 3$ [1]:

- $\mathrm{disc}(n, 3) = 3/2$ for all even $n \geq 8$ (the exception is $\mathrm{disc}(6, 3) = 1/2$),
- $\mathrm{disc}(6t \pm 1, 3) = 2$ (except for $\mathrm{disc}(5, 3) = 1$),
- $\mathrm{disc}(6t + 3, 3) \leq 2$ if $t \geq 2$, $\mathrm{disc}(9, 3) = 1$.

Morris [9] and Stefanović [2] determined exact values of $\mathrm{msum}(n, k)$ for small n, k and obtained various bounds. Determination of $\mathrm{msum}(n, 3)$ was a task at programming contest [10]. Rolfe [3], [4] suggested to use the determination of $\mathrm{msum}(n, 3)$ as a replacement of 8-queens problem while teaching students backtracking. An exercise in the discrete math text by Liu [7] is to prove that $\mathrm{msum}(36, 3) \geq 1/2$. In [1], [8] the simpler problem of linear discrepancy is considered, restricting to linear permutations (not reducing indices modulo n). Smallest value of maximum sum of two subsequent terms of permutations is considered in [5], with the additional constraint that sums are distinct. Erdös et all [6] give the asymptotic upper bound on the minimal maximum least common multiple of subsequent terms of a permutation.

Here we prove that if $t \geq 2$, then $\mathrm{disc}(6t + 3, 3) > 1$, thus solving the remaining case for $k = 3$. Furthermore, we show that $\mathrm{msum}(2km \pm 2, 2k) = 1$.

II. Simple lower bound

In [1] the fact is used that $\mathrm{disc}(n, 3)$, being less than $3/2$ cannot be $1/2$ if $n \geq 8$. We now derive analogous simple more general lower bound on $\mathrm{msum}(n, k) \leq \mathrm{disc}(n, k)$.

Theorem 1. *If $n > 2k$, then* $\mathrm{msum}(n, k) > 1/2$.

Proof. Let $x = k(n+1)/2$. Note that if $n > k$ then $\mathrm{msum}(n, k) > 0$. Indeed, if $\pi \in S_n$ is such that $s_1 = s_2 = \cdots = s_n = x$, then $0 = d_1 = s_2 - s_1 = \pi_{k+1} - \pi_1$ and π is not a permutation.

If k is even and $n > 2k$, then $\mathrm{msum}(n, k) \geq 1 > 1/2$ is an integer. Now suppose k is odd, $n > 2k$ and let $\pi \in S_n$ be a permutation such that $\mathrm{msum}(\pi, k) = 1/2$, i.e. $s_i \leq x + 1/2$ for all i, $1 \leq i \leq n$. An equality $s_i = s_{i+1}$ would imply $\pi_{k+i} = \pi_i$, hence each two maximal sums s_i (the sums equal to $x + 1/2$) are separated by at least one smaller sum (a sum that is less than or equal to $x - 1/2$). The fact that the maximal and smaller sums are interleaved imply that if the number of

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maximal sums is m, then the number $n-m$ of smaller sums is at least m. The opposite inequality $n-m \leq m$ follows from the fact that if $n-m > m$, then the average of all s_i would be less than

$$\begin{aligned} & m(x+1/2)/n + (n-m)(x-1/2)/n \\ = \quad & x + (2n-m)/2 < x \end{aligned}$$

Therefore $n-m = m$, and the smaller sums are all equal to $x-1/2$. If for example $s_2 - s_1 = -1$, then $\pi_{i+k} - \pi_i = s_{i+1} - s_i = (-1)^i/2$, $i \geq 1$, and $\pi_{2k+1} = \pi_{k+1} + d_{k+1} = \pi_{k+1} + d_{k+1} + d_1 = \pi_1$. This is a contradiction, because $2k+1 \leq n$. □

If k is odd, then $\mathrm{disc}(2kt, k)$ is not an integer. Combining $\mathrm{disc}(2kt, k) \leq 2$ [1][Theorem 9] and $\mathrm{disc}(2kt, k) > 1/2$, we obtain

Corrolary 2. *If k is odd and $t > 1$, then* $\mathrm{disc}(2kt, k) = \mathrm{msum}(2kt, k) = 3/2$.

III. Determination of $\mathrm{disc}(6t+3, 3)$

Similar, but more elaborate consideration gives the lower bound on $\mathrm{disc}(6t+3, 3)$, equal to the upper bound from [1].

Theorem 3. *Let $n = 6t+3 \geq 15$. Then* $\mathrm{disc}(n, 3) = 2$.

Proof. By [1][Theorem 4] $\mathrm{disc}(n, 3) \leq 2$. It remains to prove that if $n = 6t+3 \geq 15$, then there is no permutation $\pi \in S_n$ such that $\mathrm{disc}(\pi, 3) = 1$. Suppose that there is such a permutation π. Let $x = 3(n+1)/2$ denote average 3-sum of π. Then $s_i \in \{x, x \pm 1\}$. For fixed m it is not hard to find all vectors $(s_1, s_2, \ldots, s_m)$, $m \leq n-3$, satisfying

$$\sum_{j=0}^{q-1} d_{p+3j} = \pi_{p+3q} - \pi_p \neq 0, \quad p, q \geq 1. \tag{5}$$

If $q = 1$ then this condition becomes $\pi_{i+3} - \pi_i = s_{i+1} - s_i \neq 0$, $1 \leq i < m$. We consider the vectors $(s_1, s_2, \ldots, s_m)$ in lexicographically decreasing order, excluding those with some two subsequent components equal. Without loss of generality, suppose $s_1 = x+1$. Table I demonstrates the determination of 11 solutions $(s_1 - x, s_2 - x, \ldots, s_m - x)$ of (5) if $m = 5$. For example, the first row corresponds to the sequence $(x+1, x, x+1, x, x+1)$, where $d_1 = s_2 - s_1 = -1$, $d_4 = s_5 - s_4 = 1$, $d_1 + d_4 = 0$; hence this vector is not a solution of (5). The longer Table III in the Appendix demonstrates the determination of all 8 solutions $(s_1 - x, s_2 - x, \ldots, s_m - x)$ of (5) if $m = 15$. In Table III the values 1 and -1 of $s_i - x$ are written as + and −, respectively.

TABLE I
Determination of 11 solutions $(s_1 - x, s_2 - x, \ldots, s_5 - x)$ of (5) for $m = 5$.

1	2	3	4	5	
1	0	1	0	1	$d_1 + d_4 = 0$
1	0	1	0	-1	solution 1
1	0	1	-1	1	solution 2
1	0	1	-1	0	$d_1 + d_4 = 0$
1	0	-1	1	0	solution 3
1	0	-1	1	-1	solution 4
1	0	-1	0	1	$d_1 + d_4 = 0$
1	0	-1	0	-1	solution 5
1	-1	1	0	1	solution 6
1	-1	1	0	-1	solution 7
1	1	1	-1	1	$d_1 + d_4 = 0$
1	-1	1	-1	0	solution 8
1	-1	0	1	0	solution 9
1	-1	0	1	-1	solution 10
1	1	0	-1	1	$d_1 + d_4 = 0$
1	-1	0	-1	0	solution 11

Let $A = (0, -1, 1)^{q-2}$, $B = ((1, -1, 0)^{q-2}$, where exponential notation denotes repetition. The proof by induction that if $m = 3q$, then the 8 solutions of (5) are given by

$$1, 0, 1, A, 0, -1, 1 \tag{6}$$
$$1, 0, 1, A, 0, -1, 0 \tag{7}$$
$$1, A, 0, -1, 1, 0, -1 \tag{8}$$
$$1, -1, 1, A, 0, -1, 1 \tag{9}$$
$$1, -1, 1, A, 0, -1, 0 \tag{10}$$
$$1, -1, 0, B, 1, 0, 1 \tag{11}$$
$$1, -1, 0, B, 1, -1, 1 \tag{12}$$
$$1, -1, 0, B, 1, -1, 0 \tag{13}$$

is given in Table II. The 8 solutions of (5) of the length $m = 3q$ of the form (6)-(13)(starting from $m = 15$) lead to 9, 7, 8 solutions of the length $m+1$, $m+2$, $m+3$, respectively. Furthermore, all solutions of the length $m+3$ are of the form (6)-(13).

Now consider the supposedly existing permutation $\pi \in S_n$, $n = 6t+3$, such that $\mathrm{disc}(\pi, 3) = 1$. The first $n - 3 = 6t$ elements $s_1 - x, s_2 - x, \ldots, s_{n-3} - x$ have one of eight forms (6)-(13), determining $d_i = s_{i+1} - s_i$, $1 \leq i \leq 6t-1$. Furthermore, from $\gcd(6t+3, 3) = 3$ and $\sum_{j=0}^{2t} d_{3j+1} = 0$ it follows that the value of $d_{6t+1} = -\sum_{j=0}^{2t-1} d_{3j+1}$ in eight cases (6)-(13) are $q, q, q, q+1, q+1, 2q-1, 2q, 2q$ respectively. If $q \geq 1$, then in all eight cases we have $d_{6t+1} \geq q$. Combining this with the inequality $d_{6t+1} = s_{6t+2} - s_{6t+1} \leq 2$, we obtain $q \leq 2$, i.e. $n \leq 15$. Therefore, if $n = 6t+3 > 15$ then there is no permutation $\pi \in S_n$ such that $\mathrm{disc}(\pi, 3) = 1$, hence $\mathrm{disc}(\pi, 3) > 1$. Explicit calculation gives $\mathrm{disc}(15, 3) = 1$. □

IV. Determination of $\mathrm{msum}(2km \pm 2, 2k)$

Theorem 4. *If $m > 2$, then* $\mathrm{msum}(2km \pm 2, 2k) = 1$.

Proof. Suppose first $n = 2km + 2$. Consider the permutation $\pi \in S_n$ given by

$$(\pi_{2k(i-1)+2j-1}, \pi_{2k(i-1)+2j}) = \begin{cases} ((k-j)m + i, (k+j)m - i + 3), & 1 \leq j \leq k,\ 1 \leq i \leq m \\ (km+1, km+2), & j = 1,\ i = m+1 \end{cases}$$

Let $x = k(2km+3)$ be an average k-sum in π. It is easily verified that $d = ((1, -1)^{km}, -km, km)$, $s_1 = s_3 = \cdots =$

TABLE II
PROOF BY INDUCTION THAT ALL SOLUTIONS OF (5) ARE GIVEN BY (6)-(13).

	$(s_1 - x, \ldots, s_{m+3} - x)$	
m	1, 0, 1, A, 0, -1, 1,	solution m,1
m+1	1, 0, 1, A, 0, -1, 1, 0	solution m+1,1
m+2	1, 0, 1, A, 0, -1, 1, 0, 1	$d_{m-2} + d_{m+1} = 0$
m+2	1, 0, 1, A, 0, -1, 1, 0, -1	solution m+2,1
m+3	1, 0, 1, A, 0, -1, 1, 0, -1, 1	solution m+3,1
m+3	1, 0, 1, A, 0, -1, 1, 0, -1, 0	solution m+3,2
m+1	1, 0, 1, A, 0, -1, 1, -1	solution m+1,2
m+2	1, 0, 1, A, 0, -1, 1, -1, 1	$d_{m-5} + d_{m-2} + d_{m+1} = 0$
m+2	1, 0, 1, A, 0, -1, 1, -1, 0	$d_{m-2} + d_{m+1} = 0$
m	1, 0, 1, A, 0, -1, 0	solution m,2
m+1	1, 0, 1, A, 0, -1, 0, 1	$d_{m-3} + d_m = 0$
m+1	1, 0, 1, A, 0, -1, 0, -1	solution m+1,3
m+2	1, 0, 1, A, 0, -1, 0, -1, 1	$d_{m-5} + d_{m-2} + d_{m+1} = 0$
m+2	1, 0, 1, A, 0, -1, 0, -1, 0	$d_{m-2} + d_{m+1} = 0$
m	1, A, 0, -1, 1, 0, -1	solution m,3
m+1	1, A, 0, -1, 1, 0, -1, 1	solution m+1,4
m+2	1, A, 0, -1, 1, 0, -1, 1, 0	solution m+2,2
m+3	1, A, 0, -1, 1, 0, -1, 1, 0, 1	$d_{m-1} + d_{m+2} = 0$
m+3	1, A, 0, -1, 1, 0, -1, 1, 0, -1	solution m+3,3
m+2	1, A, 0, -1, 1, 0, -1, 1, -1	solution m+2,3
m+3	1, A, 0, -1, 1, 0, -1, 1, -1, 1	$d_{m-4} + d_{m-1} + d_{m+2} = 0$
m+3	1, A, 0, -1, 1, 0, -1, 1, -1, 0	$d_{m-1} + d_{m+2} = 0$
m+1	1, A, 0, -1, 1, 0, -1, 0	solution m+1,5
m+2	1, A, 0, -1, 1, 0, -1, 0, 1	$d_{m-2} + d_{m+1} = 0$
m+2	1, A, 0, -1, 1, 0, -1, 0, -1	solution m+2,4
m+3	1, A, 0, -1, 1, 0, -1, 0, -1, 1	$d_{m-4} + d_{m-1} + d_{m+2} = 0$
m+3	1, A, 0, -1, 1, 0, -1, 0, -1, 0	$d_{m-1} + d_{m+2} = 0$
m	1, A, 0, -1, 1, -1, 1	$d_8 + d_{m-4} + d_{m-1}$
m	1, -1, 1, A, 0, -1, 1	solution m,4
m+1	1, -1, 1, A, 0, -1, 1, 0	solution m+1,6
m+2	1, -1, 1, A, 0, -1, 1, 0, 1	$d_{m-2} + d_{m+1} = 0$
m+2	1, -1, 1, A, 0, -1, 1, 0, -1	solution m+2,5
m+3	1, -1, 1, A, 0, -1, 1, 0, -1, 1	solution m+3,4
m+3	1, -1, 1, A, 0, -1, 1, 0, -1, 0	solution m+3,5
m+1	1, -1, 1, A, 0, -1, 1, -1	solution m+1,7
m+2	1, -1, 1, A, 0, -1, 1, -1, 1	$d_{m-5} + d_{m-2} + d_{m+1} = 0$
m+2	1, -1, 1, A, 0, -1, 1, -1, 0	$d_{m-2} + d_{m+1} = 0$
m	1, -1, 1, A, 0, -1, 0	solution m,5
m+1	1, -1, 1, A, 0, -1, 0, 1	$d_{m-3} + d_m = 0$
m+1	1, -1, 1, A, 0, -1, 0, -1	solution m+1,8
m+2	1, -1, 1, A, 0, -1, 0, -1, 1	$d_{m-5} + d_{m-2} + d_{m+1} = 0$
m+2	1, -1, 1, A, 0, -1, 0, -1, 0	$d_{m-2} + d_{m+1} = 0$
m	1, -1, 0, B, 1, 0, 1	solution m,6
m+1	1, -1, 0, B, 1, 0, 1, 0	$d_{m-3} + d_m = 0$
m+1	1, -1, 0, B, 1, 0, 1, -1	$d_9 + d_{m-3} + d_m = 0$
m	1, -1, 0, B, 1, -1, 1	solution m,7
m+1	1, -1, 0, B, 1, -1, 1, 0	$d_{m-3} + d_m = 0$
m+1	1, -1, 0, B, 1, -1, 1, -1	$d_9 + d_{m-3} + d_m = 0$
m	1, -1, 0, B, 1, -1, 0	solution m,7
m+1	1, -1, 0, B, 1, -1, 0, 1	solution m+1,9
m+2	1, -1, 0, B, 1, -1, 0, 1, 0	solution m+2,6
m+3	1, -1, 0, B, 1, -1, 0, 1, 0, 1	solution m+3,6
m+3	1, -1, 0, B, 1, -1, 0, 1, 0, -1	$d_{m-1} + d_{m+2} = 0$
m+2	1, -1, 0, B, 1, -1, 0, 1, -1	solution m+2,7
m+3	1, -1, 0, B, 1, -1, 0, 1, -1, 1	solution m+3,7
m+3	1, -1, 0, B, 1, -1, 0, 1, -1, 0	solution m+3,8
m+1	1, -1, 0, B, 1, -1, 0, -1	$d_{m-3} + d_m = 0$

TABLE III
PROOF THAT IF $m = 15$, THEN THE 8 SOLUTIONS OF (5) ARE (6)-(13).

m	$s_1 - x, s_2 - x, \ldots, s_m - x$	comment
5	+ 0 + 0 +	$d_1 + d_4 = 0$
8	+ 0 + 0 − + 0 +	$d_4 + d_7 = 0$
11	+ 0 + 0 − + 0 − + 0 +	$d_7 + d_{10} = 0$
14	+ 0 + 0 − + 0 − + 0 − + 0 +	$d_{10} + d_{13} = 0$
15	+ 0 + 0 − + 0 − + 0 − + 0 − +	solution 15, 1
15	+ 0 + 0 − + 0 − + 0 − + 0 − 0	solution 15, 2
14	+ 0 + 0 − + 0 − + 0 − + − +	$d_7 + d_{10} + d_{13} = 0$
14	+ 0 + 0 − + 0 − + 0 − + − 0	$d_{10} + d_{13} = 0$
13	+ 0 + 0 − + 0 − + 0 − 0 +	$d_9 + d_{12} = 0$

$s_{2km} = x - 1/2$, $s_2 = s_4 = \cdots = s_{2km-1} = x + 1/2$, and $s_{2km+2} = x - km - 1/2$, hence $\text{msum}(\pi, 2k) = 1$.

Suppose now $n = 2km - 2$. Let the permutation $\pi \in S_n$ is defined by

$$(\pi_{2k(i-1)+2j-1}, \pi_{2k(i-1)+2j} =$$
$$\begin{cases} ((i, 2km - i - 1), & j = k,\ 1 \leq i \leq m \\ (km - 1, km), & j = k - 1,\ i = m \\ (jm + i - 1, (2k - j)m - i), & \text{otherwise} \end{cases}$$

The average k-sum in π is now $x = k(2km - 1)$. The vector d is $((1, -1)^{km-2}, -km + 2, km - 2)$, implying $s_1 = s_3 = \cdots = s_{2km-3} = x - 1/2$, $s_2 = s_4 = \cdots = s_{2km-4} = x + 1/2$, $s_{2km+2} = x - km + 3/2$ and $\text{disc}(\pi, 2k) = 1$. $\square$

From [1][Theorem 9] it follows $\text{disc}(2kt \pm 2, 2k) \leq 2$, while by Theorem 4 $\text{msum}(2kt \pm 2, 2k) = 1$. In many cases $\text{disc}(2kt \pm 2, 2k)$ is strictly greater than $\text{msum}(2kt \pm 2, 2k)$. Explicit computation in [2] by backtracking algorithm similar to the one from [4] shows that $\text{disc}(4t + 2, 4) = 2$ if $10 \leq 4t+2 \leq 54$; $\text{disc}(6t \pm 2, 6) = 2$ if $10 \leq n = 6t \pm 2 \leq 38$; $\text{disc}(8t \pm 2, 8) = 2$ if $14 \leq n = 6t \pm 2 \leq 30$.

APPENDIX

Proof that if $m = 15$, then the 8 solutions of (5) are (6)-(13) is given in Table III.

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TABLE III
(CONTINUED)

m	$s_1 - x, s_2 - x, \ldots, s_m - x$	comment
14	+ 0 + 0 − + 0 − + 0 − 0 − +	$d_7 + d_{10} + d_{13} = 0$
14	+ 0 + 0 − + 0 − + 0 − 0 − 0	$d_{10} + d_{13} = 0$
11	+ 0 + 0 − + 0 − + − +	$d_4 + d_7 + d_{10} = 0$
11	+ 0 + 0 − + 0 − + − 0	$d_7 + d_{10} = 0$
10	+ 0 + 0 − + 0 − 0 +	$d_6 + d_9 = 0$
11	+ 0 + 0 − + 0 − 0 − +	$d_4 + d_7 + d_{10} = 0$
11	+ 0 + 0 − + 0 − 0 − 0	$d_7 + d_{10} = 0$
8	+ 0 + 0 − + − +	$d_1 + d_4 + d_7 = 0$
8	+ 0 + 0 − + − 0	$d_4 + d_7 = 0$
7	+ 0 + 0 − 0 +	$d_3 + d_6 = 0$
8	+ 0 + 0 − 0 − +	$d_1 + d_4 + d_7 = 0$
8	+ 0 + 0 − 0 − 0	$d_4 + d_7 = 0$
6	+ 0 + − + 0	$d_2 + d_5 = 0$
7	+ 0 + − + − +	$d_3 + d_6 = 0$
10	+ 0 + − + − 0 + 0 +	$d_3 + d_6 + d_9 = 0$
10	+ 0 + − + − 0 + 0 −	$d_6 + d_9 = 0$
11	+ 0 + − + − 0 + − + 0	$d_7 + d_{10} = 0$
11	+ 0 + − + − 0 + − + −	$d_1 + d_4 + d_7 + d_{10} = 0$
10	+ 0 + − + − 0 + − 0	$d_3 + d_6 + d_9 = 0$
8	+ 0 + − + − 0 −	$d_1 + d_4 + d_7 = 0$
5	+ 0 + − 0	$d_1 + d_4 = 0$
6	+ 0 − + 0 +	$d_2 + d_5 = 0$
9	+ 0 − + 0 − + 0 +	$d_5 + d_8 = 0$
12	+ 0 − + 0 − + 0 − + 0 +	$d_8 + d_{11} = 0$
15	+ 0 − + 0 − + 0 − + 0 − + 0 +	$d_{11} + d_{14} = 0$
15	+ 0 − + 0 − + 0 − + 0 − + 0 −	solution 15, 3
15	+ 0 − + 0 − + 0 − + 0 − + − +	$d_8 + d_{11} + d_{14} = 0$
15	+ 0 − + 0 − + 0 − + 0 − + − 0	$d_{11} + d_{14} = 0$
14	+ 0 − + 0 − + 0 − + 0 − 0 +	$d_{10} + d_{13} = 0$
15	+ 0 − + 0 − + 0 − + 0 − 0 − +	$d_8 + d_{11} + d_{14} = 0$
15	+ 0 − + 0 − + 0 − + 0 − 0 − 0	$d_{11} + d_{14} = 0$
12	+ 0 − + 0 − + 0 − + − +	$d_5 + d_8 + d_{11} = 0$
12	+ 0 − + 0 − + 0 − + − 0	$d_8 + d_{11} = 0$
11	+ 0 − + 0 − + 0 − 0 +	$d_7 + d_{10} = 0$
12	+ 0 − + 0 − + 0 − 0 − +	$d_5 + d_8 + d_{11} = 0$
12	+ 0 − + 0 − + 0 − 0 − 0	$d_8 + d_{11} = 0$
9	+ 0 − + 0 − + − +	$d_2 + d_5 + d_8 = 0$
9	+ 0 − + 0 − + − 0	$d_5 + d_8 = 0$
8	+ 0 − + 0 − 0 +	$d_4 + d_7 = 0$
9	+ 0 − + 0 − 0 − +	$d_2 + d_5 + d_8 = 0$
9	+ 0 − + 0 − 0 − 0	$d_5 + d_8 = 0$
9	+ 0 − + − + 0 + 0	$d_2 + d_5 + d_8 = 0$
9	+ 0 − + − + 0 + −	$d_5 + d_8 = 0$
10	+ 0 − + − + 0 − + 0	$d_3 + d_6 + d_9 = 0$
13	+ 0 − + − + 0 − + − + 0 +	$d_3 + d_6 + d_9 + d_{12} = 0$
14	+ 0 − + − + 0 − + − + 0 − +	$d_1 + d_4 + \ldots + d_{13} = 0$
14	+ 0 − + − + 0 − + − + 0 − 0	$d_4 + d_7 + d_{10} + d_{13} = 0$
12	+ 0 − + − + 0 − + − + −	$d_8 + d_{11} = 0$
11	+ 0 − + − + 0 − + − 0	$d_7 + d_{10} = 0$
10	+ 0 − + − + 0 − 0 +	$d_6 + d_9 = 0$
10	+ 0 − + − + 0 − 0 −	$d_3 + d_6 + d_9 = 0$
7	+ 0 − + − + −	$d_3 + d_6 = 0$
6	+ 0 − + − 0	$d_2 + d_5 = 0$
5	+ 0 − 0 +	$d_1 + d_4 = 0$
7	+ 0 − 0 − + 0	$d_3 + d_6 = 0$
8	+ 0 − 0 − + − +	$d_1 + d_4 + d_7 = 0$
8	+ 0 − 0 − + − 0	$d_4 + d_7 = 0$
6	+ 0 − 0 − 0	$d_2 + d_5 = 0$
7	+ − + 0 + 0 +	$d_3 + d_6 = 0$
9	+ − + 0 + 0 − + 0	$d_2 + d_5 + d_8 = 0$
10	+ − + 0 + 0 − + − +	$d_3 + d_6 + d_9 = 0$
10	+ − + 0 + 0 − + − 0	$d_6 + d_9 = 0$
8	+ − + 0 + 0 − 0	$d_1 + d_4 + d_7 = 0$
6	+ − + 0 + −	$d_2 + d_5 = 0$
8	+ − + 0 − + 0 +	$d_4 + d_7 = 0$
11	+ − + 0 − + 0 − + 0 +	$d_7 + d_{10} = 0$
14	+ − + 0 − + 0 − + 0 − + 0 +	$d_{10} + d_{13} = 0$
15	+ − + 0 − + 0 − + 0 − + 0 − +	solution 15, 4
15	+ − + 0 − + 0 − + 0 − + 0 − 0	solution 15, 5
14	+ − + 0 − + 0 − + 0 − + − +	$d_7 + d_{10} + d_{13} = 0$
14	+ − + 0 − + 0 − + 0 − + − 0	$d_{10} + d_{13} = 0$
13	+ − + 0 − + 0 − + 0 − 0 +	$d_9 + d_{12} = 0$
14	+ − + 0 − + 0 − + 0 − 0 − +	$d_7 + d_{10} + d_{13} = 0$
14	+ − + 0 − + 0 − + 0 − 0 − 0	$d_{10} + d_{13} = 0$

TABLE III
(CONTINUED)

m	$s_1 - x, s_2 - x, \ldots, s_m - x$	comment
11	+ − + 0 − + 0 − + − +	$d_4 + d_7 + d_{10} = 0$
11	+ − + 0 − + 0 − + − 0	$d_7 + d_{10} = 0$
10	+ − + 0 − + 0 − 0 +	$d_6 + d_9 = 0$
11	+ − + 0 − + 0 − 0 − +	$d_4 + d_7 + d_{10} = 0$
11	+ − + 0 − + 0 − 0 − 0	$d_7 + d_{10} = 0$
11	+ − + 0 − + − + 0 + 0	$d_4 + d_7 + d_{10} = 0$
11	+ − + 0 − + − + 0 + −	$d_7 + d_{10} = 0$
12	+ − + 0 − + − + 0 − + 0	$d_5 + d_8 + d_{11} = 0$
14	+ − + 0 − + − + 0 − + − + 0	$d_1 + d_4 + \ldots + d_{13} = 0$
14	+ − + 0 − + − + 0 − + − + −	$d_{10} + d_{13} = 0$
13	+ − + 0 − + − + 0 − + − 0	$d_9 + d_{12} = 0$
11	+ − + 0 − + − + 0 − 0	$d_1 + d_4 + d_7 + d_{10} = 0$
9	+ − + 0 − + − + −	$d_5 + d_8 = 0$
8	+ − + 0 − + − 0	$d_4 + d_7 = 0$
7	+ − + 0 − 0 +	$d_3 + d_6 = 0$
9	+ − + 0 − 0 − + 0	$d_5 + d_8 = 0$
10	+ − + 0 − 0 − + − +	$d_3 + d_6 + d_9 = 0$
10	+ − + 0 − 0 − + − 0	$d_6 + d_9 = 0$
8	+ − + 0 − 0 − 0	$d_4 + d_7 = 0$
5	+ − + − +	$d_1 + d_4 = 0$
8	+ − + − 0 + 0 +	$d_1 + d_4 + d_7 = 0$
8	+ − + − 0 + 0 −	$d_4 + d_7 = 0$
9	+ − + − 0 + − + 0	$d_5 + d_8 = 0$
10	+ − + − 0 + − + − +	$d_6 + d_9 = 0$
12	+ − + − 0 + − + − 0 + 0	$d_2 + d_5 + d_8 + d_{11} = 0$
14	+ − + − 0 + − + − 0 + − + 0	$d_{10} + d_{13} = 0$
14	+ − + − 0 + − + − 0 + − + −	$d_1 + d_4 + \ldots + d_{13} = 0$
13	+ − + − 0 + − + − 0 + − 0	$d_6 + d_9 + d_{12} = 0$
11	+ − + − 0 + − + − 0 −	$d_1 + d_4 + d_7 + d_{10} = 0$
8	+ − + − 0 + − 0	$d_1 + d_4 + d_7 = 0$
7	+ − + − 0 − +	$d_3 + d_6 = 0$
8	+ − + − 0 − 0 +	$d_1 + d_4 + d_7 = 0$
8	+ − + − 0 − 0 −	$d_4 + d_7 = 0$
7	+ − 0 + 0 + 0	$d_3 + d_6 = 0$
9	+ − 0 + 0 + − + 0	$d_5 + d_8 = 0$
9	+ − 0 + 0 + − + −	$d_2 + d_5 + d_8 = 0$
8	+ − 0 + 0 + − 0	$d_4 + d_7 = 0$
6	+ − 0 + 0 −	$d_2 + d_5 = 0$
7	+ − 0 + − + 0	$d_3 + d_6 = 0$
8	+ − 0 + − + − +	$d_4 + d_7 = 0$
11	+ − 0 + − + − 0 + 0 +	$d_4 + d_7 + d_{10} = 0$
11	+ − 0 + − + − 0 + 0 −	$d_7 + d_{10} = 0$
12	+ − 0 + − + − 0 + − + 0	$d_8 + d_{11} = 0$
13	+ − 0 + − + − 0 + − + − +	$d_9 + d_{12} = 0$
14	+ − 0 + − + − 0 + − + − 0 +	$d_1 + d_4 + \ldots + d_{13} = 0$
14	+ − 0 + − + − 0 + − + − 0 −	$d_4 + d_7 + d_{10} + d_{13} = 0$
11	+ − 0 + − + − 0 + − 0	$d_4 + d_7 + d_{10} = 0$
10	+ − 0 + − + − 0 − +	$d_6 + d_9 = 0$
10	+ − 0 + − + − 0 − 0	$d_3 + d_6 + d_9 = 0$
10	+ − 0 + − 0 + 0 + 0	$d_6 + d_9 = 0$
10	+ − 0 + − 0 + 0 + −	$d_3 + d_6 + d_9 = 0$
9	+ − 0 + − 0 + 0 −	$d_5 + d_8 = 0$
10	+ − 0 + − 0 + − + 0	$d_6 + d_9 = 0$
10	+ − 0 + − 0 + − + −	$d_3 + d_6 + d_9 = 0$
13	+ − 0 + − 0 + − 0 + 0 + 0	$d_9 + d_{12} = 0$
13	+ − 0 + − 0 + − 0 + 0 + −	$d_6 + d_9 + d_{12} = 0$
12	+ − 0 + − 0 + − 0 + 0 −	$d_8 + d_{11} = 0$
13	+ − 0 + − 0 + − 0 + − + 0	$d_9 + d_{12} = 0$
13	+ − 0 + − 0 + − 0 + − + −	$d_6 + d_9 + d_{12} = 0$
15	+ − 0 + − 0 + − 0 + − 0 + 0 +	solution 15, 6
15	+ − 0 + − 0 + − 0 + − 0 + 0 −	$d_{11} + d_{14} = 0$
15	+ − 0 + − 0 + − 0 + − 0 + − +	solution 15, 7
15	+ − 0 + − 0 + − 0 + − 0 + − 0	solution 15, 8
13	+ − 0 + − 0 + − 0 + − 0 −	$d_9 + d_{12} = 0$
10	+ − 0 + − 0 + − 0 −	$d_6 + d_9 = 0$
7	+ − 0 + − 0 −	$d_3 + d_6 = 0$
5	+ − 0 − +	$d_1 + d_4 = 0$
8	+ − 0 − 0 + 0 +	$d_1 + d_4 + d_7 = 0$
8	+ − 0 − 0 + 0 −	$d_4 + d_7 = 0$
9	+ − 0 − 0 + − + 0	$d_5 + d_8 = 0$
9	+ − 0 − 0 + − + −	$d_2 + d_5 + d_8 = 0$
8	+ − 0 − 0 + − 0	$d_1 + d_4 + d_7 = 0$
6	+ − 0 − 0 −	$d_2 + d_5 = 0$